Math 254B Lecture 26 Notes

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1 Intersections of Lines with Fractals and Introduction to Scenery Processes

1.1 Intersection of lines with fractals

We have iterated function systems in $\mathbb{R}^2$ with $\Phi_i(x) = rUx + a_o$ fpr $1 \leq i \leq k$, where U is rotation by $2\pi\xi$ for $\xi \in \mathbb{R} \setminus \mathbb{Q}$. We are assuming the SSC, so $\pi : [k]^{\mathbb{N}} \to K$ is a conjugacy from $\sigma \to S$, where K is the attractor of the system. We denote $K_i = \Phi_i[K]$ and $K_w = \Phi_w[K]$ for words w.

If $D \supseteq K$, then $D_w \supseteq K_w$ for all w Then $K = \bigcap_{n \geq 1} \bigcup_{w \in [k]^n} D_w$.

Given $z \in \mathbb{R}^2$ and $u \in S^1$, write $L_{z,u}$ for the line through z parallel to u. We will prove this result.

Theorem 1.1. *Fix $L \in \mathbb{R}^2$. For a.e. $u \in S^1$, there is a $z \in K$ such that*

$$\dim(K \cap L_{z,u}) \geq \dim(K \cap L).$$

The machinery we develop will allow us to prove:

Theorem 1.2. $\dim(K \cap L) \leq \max\{0, \dim(K) - 1\}$ *for all lines L.*

Notation: Let $P(X)$ be the set of probability measures on X. If $Y \in \mathcal{B}_X$, then $P(Y) \subseteq P(X)$. If X is a compact metric space and $Y \in \mathcal{B}_X$, then $P(Y) \in \mathcal{B}_{P(X)}$ for the weak* topology.[1]

Definition 1.1. If $\mu \in P(P)$, we say μ is **carried by** Y.

Definition 1.2. If $\mu \in P(X)$ and X is a compact metric space, let

$$W = \{x \in X : \mu(U) = 0 \text{ for some neighborhood } U \ni x\}.$$

$Y := X \setminus W$ is called the **support** of μ.

[1] This is an exercise in the monotone class theorem. Professor Austin says that he has never met someone who enjoys the monotone class theorem, but this is actually false. I like the monotone class theorem!

Then $\mu(Y) = 1$, and

$$Y = \{x \in X : \mu(B_r(x)) > 0 \,\forall t > 0\}.$$

If we have a "big" intersection of a line with K in some direction u, by pushing the dynamics forward, we get "big" intersections in all directions ofr the form $u_0 e^{-2\pi i m\xi}$:

$$\dim(K \cap L) = \max_i \dim(K_i \cap L) = \max_i \dim(\underbrace{S(K_i \cap L)}_{=K \cap \Phi_i^{-1}(L)}).$$

Hope: We want good intersections in a direction $u \in S^1$. Maybe we can find lines $L^{(1)}, L^{(2)}, \dots$ with directions $u_1, u_2, \dots, \in S^1$ such that $u_n \to u$. Then $L^{(n_i)}$ converges to some limit line L in direction u. So $\dim(K \cap L) = \lim_i \dim(K \cap L^{(u_i)})$. However, this doesn't work. Hausdorff dimension is incredibly discontinuous. We have to deal with this.[2]

1.2 Scenery dynamics with probability measures

Let $\alpha : K \to [k]$, $\alpha_{[1;n]} : K \to [k]^n$, and $\alpha_n : K \to [k]$. Denote $[z]_1 = \{z' : \alpha_1(z') = \alpha_1(z)\}$, and $[z]_1^n = \{z' : \alpha_{[1,n]}(z') = \alpha_{[1,n]}(z)\}$. We want to define

$$T_0(z, \nu) = (Sz, S\left(\frac{\nu(\cdot \cap [z])}{\nu([z]_1)}\right)).$$

This is defined only on $U = \{(z, \nu) : \nu([z]_1) > 0\}$. So this is $T_0 : U \to K \times P(K)$. We need to restrict to $X = \bigcap_{n \geq 1} T_0^{-n}[U]$ and let $T = T_0|_X$.

Definition 1.3. (X, T) is the **CP system**.

We want to use Bogliubov-Krylov in this setting to product invariant distributions on the space of probability measures on $K \times P(K)$.

Lemma 1.1. $X = \{(z, \nu) : z \in \operatorname{supp}(\nu)\}$.

Proof. $(z, \nu) \in T_0^{-n}[U] \iff \nu([z]_1) > 0$ and $S_* \nu_{[z]_1}([Sz]_1) = \nu_{[z_1]}([z]_2) > 0$ and so on to say $\nu([z]_1^n) > 0$. This is equivalent to $\nu([z]_{1,2}) > 0$. □

Notation: If $\nu(K_w) > 0$, then $\nu^w = S_*^n(\nu|_{K_w})$.

These is a special subclass in $P(L \times P(K))$.

Definition 1.4. If $\widehat{\mu} \in P(K \times P(K))$ has second marginal $\overline{\mu}$, $\widehat{\mu}$ is **adapted** if

$$\widehat{\mu} = \int_{P(K)} \nu \times \delta_\nu \, d\overline{\mu}(\nu).$$

[2] Analysis is the type of subject where you have to roll up your sleeves and walk into the jungle.

In other words, choosing a random pair (z, ν) using $\widehat{\mu}$ is the same as choosing ν ccording to $\overline{\mu}$ and then choosing z according to ν.

Lemma 1.2. *If $\widehat{\mu}$ is adapted, the $\widehat{\mu}(X) = 1$.*

Proof.

$$\widehat{\mu}(X) = \int (\nu \times \delta_\nu) * X(\, d\overline{\mu}(\nu) = \int \nu(\mathrm{supp}(\nu)) \, d\overline{\mu}(\nu) = 1. \qquad \square$$

Let us rewrite the definition of boing adapted in the following way: $\widehat{\mu}$ is adapted iff $f : K \times P(K) \to \mathbb{R}$ by

$$\int f(z,\nu) \, d\widehat{\mu}(z,\nu) = \int_{P(K)} \underbrace{\left[\int_K f(z,\nu) \, d\nu(z) \right]}_{Qf(z,\nu)} d\overline{\mu}(\nu).$$

The function $Qf(z, \nu)$ does not actually depend on z, but we want to think of it as a function on the same space.

Lemma 1.3. *$\widehat{\mu}$ is adapted if and only if*

$$\int f \, d\widehat{\mu} = \int Qf \, d\widehat{\mu} \qquad \forall f \in C(K \times P(K)).$$

Lemma 1.4. *Q defines a bounded operator $C(K \times P(K)) \to C(K \times P(K))$.*

Proof. We need to show that if f is continuous, Qf is continuous. First, let $f(z,\nu) = f_1(z) f_2(\nu)$. Then $Qf(z,\nu) = (\int f_1 \, d\nu) \cdot f_2(\nu)$, so $Q(fz, \nu)$. By Stone-Weierstrass, a continuous function can be uniformly approximated by functions of the aforementioned form. Now use $\|Qf\|_u \leq \|f\|_u$. $\square$

Corollary 1.1. *The set P_a of adapted distributions is a weak*-closed subset of $P(K \times P(K))$.*

Proof. Observe that

$$P_a = \bigcap_{P \in C(K \times P(K))} \{\widehat{\mu} : \int (f - Qf) \, d\widehat{\mu} = 0\}.$$

This is an intersection of vanishing sets of continuous functions. $\square$

Remark 1.1. P_a is also convex.

Proposition 1.1. *$T_* : P_a \to P_a$ is continuous.*

This follows form the following:

Lemma 1.5. *If $\widehat{\mu} \in P_a$, then $T_*\widehat{\mu} \in P_a$, and its second marginal is*

$$M\overline{\mu} = \int \sum_{i=1}^{k} \nu(K_i) \cdot \delta_{\nu^i} \, d\overline{\mu}(\nu).$$

We will prove the lemma next time.

Corollary 1.2. *If $\widehat{\mu}^{(0)} \in P_a$ and $\widehat{\mu}^{(n)} := \frac{1}{n}\sum_{i=1}^{n} T_*^i \widehat{\mu}^{(0)}$ and $\widehat{\mu}^{(n_i)} \xrightarrow{weak*} \widehat{\mu}$, then $\widehat{\mu} \in P_a$ and is T-invariant: $T_*\widehat{\mu}^{(n)} = \widehat{\mu}^{(n)} + O(1/n)$.*